The Beauty of Euler's Identity

The first time I was stunned by Euler's work was in Calculus when I was introduced to Euler's identity, $e^{(i\pi)} + 1 = 0$. I was amazed to see such a simple formula that included e, i, and π. Ever since I have been fascinated by Euler's work, the 18th-century mathematician continues to amaze me every time I see his work appearing in different branches of math such as graph theory and topology and even other subjects, for example, physics, astronomy, and music theory. The most common of his equations that I use is Euler's formula, $e^{(ix)} = cos(x) + i\ sin(x)$. It not only has an interesting proof and but also has many applications.

Although there are a few different approaches to prove that $e^{(ix)} = cos(x) + i\ sin(x)$, I will be showing a proof that utilizes power series and a basic understanding of imaginary numbers. This method highlights a simplistic and elegant proof that only adds to the beauty of Euler's formula. From the power series, we know that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. By substituting ix = z,

$$e^{(ix)} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

$$e^{(ix)} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

$$e^{(ix)} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{ix^5}{5!} - \cdots\right)$$

$$e^{(ix)} = \cos(x) + i\sin(x)$$

The last two steps rearrange the expansion into the Maclaurin series for sin(x) and cos(x) and thus Euler's formula is proven in four lines. The beauty of the formula continues when deriving Euler's identity. By simply letting x = π we get the following,

$$e^{(i\pi)} = \cos(\pi) + i\sin(\pi) \Rightarrow e^{(i\pi)} = -1 + 0 \Rightarrow e^{(i\pi)} + 1 = 0$$

And thus, Euler's formula has allowed us to derive Euler's identity. Euler's formula allows us to convert from the complex and the polar plane to the Cartesian plane and allows mathematics to derive complex trigonometric identities. These relationships allow electrical engineers to apply complex numbers in their work and has created many new questions in topology. His work is also often seen in calculus and trigonometry. Euler truly captured the beauty and the applications of mathematics in his lifetime.